The Limit of Frequency Estimation

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Abstract

In phase and frequency measurements, the measured phase and frequency are not the true phase and frequency but the ones which are disturbed by noises, due to the effects of the noise processes. In this paper, we discussed the effects of three noise processes, i.e., White PM, White FM and Random Walk FM, on the estimations of phase and frequency. It is indicated that the properties of these two estimations are very different. In phase estimation, the error can be reduced by properly selecting suitable smoothing length *NT* and smaple interval *T*. But in frequency estimation, the error cannot be reduced arbitarily by means of improving estimator or measurement equipment. The precison of frequency is limited by the intrinsic noises in the clock.

I. Introduction

In phase and frequency measurements, the measured phase and frequency are not the true phase and frequency but the disturbed ones, due to the effects of the noise processes. Before an atomic time scale being computed, the phase and frequency of each clock should be measured. So it is necessary to give proper estimations of the true phase and frequency.

In this paper, we discussed the effects of three noise processes, i.e., White PM, White FM and Random Walk FM, on the estimations of the phase and frequency. It is indicated that the properties of these two estimations are very different.

II. The Noise Model

The basic model, so called Power-law Model, is presented in frequency domain[1],

$$S_y(f) = h_{-2}f^{-2}+h_{-1}f^{-1}+h_0+h_1f+h_2f^2 \qquad (1)$$

In time domain, the Dynamic Model is more convenient[2],

$$\begin{aligned} T(k) &= x(k)+r(k) \\ x(k+1) &= x(k)+y(k)\,T+q(k) \\ y(k+1) &= y(k)+f(k) \end{aligned} \qquad (2)$$

where $r(k)$ are phase modulation noise processes which contain White PM and Flicker PM; $q(k)$ are frequency modulation noise processes including White FM, Flicker FM and Random Walk FM; and $f(k)$ is Random Walk FM. The sampling interval is T.

The two flickers are not considered in the following disscusion, because the other three noise processes in different levels are enough to demenstrate the properties of the phase and frequenbcy estimations. So $r(k)$ represents the White

PM only,

$$Dr(k)^2 = Er(k)^2 = \frac{1}{(2\pi)^2} h_2 f_h \tag{3}$$

and $q(k)$ represents the White FM and the Random Walk FM,

$$Dq(k)^2 = Eq(k)^2 = \frac{1}{2} h_0 T + \frac{1}{6} (2\pi)^2 h_{-2} T^3 \tag{4}$$

III. Phase Estimation (finite-memory smoothing)

For a set of $2N+1$ phase measurements of $T(k+i)(i=-N,\ldots,0,\ldots,N)$, the linear estimation of $x(t)$ is

$$\hat{x}(k) = \sum_{i=-N}^{N} C_i T(k+i) \tag{5}$$

There is a set of cofficients, $C_i(i=-N,\ldots,0,\ldots,N)$, which enables the estimation error to be minum,

$$E[T(k) - \hat{x}(k)]^2 \rightarrow \min \tag{6}$$

But we do not care about the optimal solution here, because the properties of the estimation are more important. For convenience, a simplified smoother, average smoother, is used here. That is,

$$C_i = \frac{1}{2N+1} \qquad i=-N,\cdots,0,\cdots,N \tag{7}$$

and

$$\begin{aligned} \hat{x}(k) &= \frac{1}{2N+1} \sum_{i=-N}^{N} T(k+i) \\ &= \frac{1}{2N+1} \sum_{i=-N}^{N} x(k+i) + \frac{1}{2N+1} \sum_{i=-N}^{N} r(k+i) \end{aligned} \tag{8}$$

If T is not large, the effect of Random Walk FM, $f(k)$, can be ignored in phase estimation. It is easy to show that

$$x(k+i) = x(k) + \sum_{l=1}^{i} q(k+l-1) \tag{9}$$

So the estimated phase can be expressed as true phase plus a serier of noise processes.

$$\begin{aligned} \hat{x}(k) &= x(k) + \frac{1}{2N+1} \sum_{i=-N}^{N} r(k+i) \\ &- \frac{1}{2N+1} \sum_{i=1}^{N} q(k-i)\,(N-i+1) \\ &+ \frac{1}{2N+1} \sum_{i=1}^{N} q(k+i-1)\,(N-i+1) \end{aligned} \tag{10}$$

According the assumption of independence between noise processes, the estimation error is

$$
\begin{aligned}
&E[x(k)-\hat{x}(k)]^2 \\
&=\frac{1}{(2N+1)}\cdot Er^2(k)+\frac{2}{(2N+1)^2}\sum_{i=1}^{N}(N-i+1)^2\cdot Eq^2(k) \\
&=\frac{1}{2N+1}\cdot\frac{1}{(2\pi)^2}h_2 f_h+\frac{N(N+1)}{2(2N+1)}h_0 T \\
&\approx\frac{1}{2N}\cdot\frac{1}{(2\pi)^2}h_2 f_h+\frac{1}{4}NTh_0
\end{aligned}
\tag{11}
$$

This is the property of phase estimation that a small NT, smoothing length, can reduce the effect of White FM and also when NT is fixed, increasing N, i.e., decreasing T, can reduce the effect of White PM. Therefore, a suitable selection of N and T will make the error of phase estimation small enough.

IV. Frequency Estimation-I (Finite-memory smoothing)

We assume the measured phase has been carefully filtered before the *frequency estimation being done.* Thus the phase noise can be ignored and the noise model of Eq.(1) is rewritten as

$$
\begin{aligned}
d(k)&=y(k)+\frac{1}{T}q(k) \\
y(k+1)&=y(k)+f(k)
\end{aligned}
\tag{12}
$$

where

$$
d(k)\triangleq\frac{1}{T}[x(k+1)-x(k)]
\tag{13}
$$

For a set of $2N+1$ frequency measurements of $d(k+i)(i=-N,\ldots,0,\ldots,N)$, the linear estimation of $y(t)$ is

$$
\hat{y}(k)=\sum_{i=-N}^{N}C_i d(k+i)
\tag{14}
$$

We also consider an average smoother, similar to the case in phase estimation. That is,

$$
\begin{aligned}
\hat{y}(k)&=\frac{1}{2N+1}\sum_{i=-N}^{N}d(k+i) \\
&=y(k)+\frac{1}{T}\cdot\frac{1}{2N+1}\sum_{i=-N}^{N}q(k+i) \\
&\quad-\frac{1}{2N+1}\sum_{i=1}^{N}f(k-i)\,(N-i+1) \\
&\quad+\frac{1}{2N+1}\sum_{i=1}^{N}f(k+i-1)\,(N-i+1)
\end{aligned}
\tag{15}
$$

The estimation error is

$$
\begin{aligned}
&E[y(k)-\hat{y}(k)]^2 \\
&=\frac{1}{(2N+1)T^2}\cdot Eq^2(k)+\frac{2}{(2N+1)^2}\sum_{i=1}^{N}(N-i+1)^2\cdot Ef^2(k) \\
&=\frac{1}{2N+1}\left[\frac{1}{2T}h_0+\frac{1}{6}(2\pi)^2h_{-2}T\right]+\frac{N(N+1)}{2N+1}\cdot\frac{1}{2}(2\pi)^2h_{-2}T \\
&\approx\frac{1}{4}NTh_0+\pi^2h_{-2}NT
\end{aligned}
\tag{16}
$$

It can be seen from Eq.(11) and Eq.(17) that estimations of phase and frequency are very different. In phase estimation, the two noise effects can be reduced simultaneously by small NT and large N. But in frequency estimation small NT may reduce the effect of h_{-2} but the effect of h_0 will increase, while large NT will lead high h_{-2} and low h_0. Therefore, the precision of frequency estimation cannot be improved arbitarily by change N or T for certain noise levels.

According to Eq.(17), we may choose a suitable NT which can make the estimation error as small as possible,

$$
(NT)_{\min}=\frac{1}{2\pi}\sqrt{\frac{h_0}{h_{-2}}}
\tag{17}
$$

Correspondent error is

$$
E[y(k)-\hat{y}(k)]^2_{\min}=\pi\sqrt{h_0h_{-2}}
\tag{18}
$$

V. Frequency Estimation-II (Kalman Filtering)

For the noise model shown in Eq.(12), Kalman filter may be considered as an optimal estimator. Obviously, the system of Eq.(12) fulfills the conditions of observability and controllability, so the filter is stable. When the system is in steady state, the frequency estimation is

$$
\hat{y}(k+1/k+1)=\hat{y}(k/k)+K[d(k+1)-\hat{y}(k/k)]
\tag{19}
$$

and the estimation error is

$$
E[y(k)-\hat{y}(k/k)]^2=P^+
\tag{20}
$$

In steady state, the filtering error P^+, the prediction error P^- and the gain K fulfill the Riccati Equation as below,

$$
\begin{aligned}
&P^-=P^++F \\
&K=P^-\left(P^-+\frac{1}{T^2}Q\right)^{-1} \\
&P^+=(1-K)P^-
\end{aligned}
\tag{21}
$$

where $F=Df^2(k)$ and $Q=D(q(k)/T)^2$. P^+ is solved as

$$
\begin{aligned}
P^+&=\frac{1}{2}\left(-F+\sqrt{F^2+4F\frac{Q}{T^2}}\right) \\
&=-\pi^2h_{-2}T+\sqrt{\pi^2h_0h_{-2}+\frac{7}{3}\pi^4h_{-2}^2T^2}
\end{aligned}
\tag{22}
$$

There is a $T_{\min}$ which makes P^+ be minium,

$$T_{\min}=\frac{1}{\pi}\sqrt{\frac{9h_0}{28h_{-2}}} \tag{23}$$

The correspondent estimation error is

$$E[y(k)-\hat{y}(k/k)]^2=\pi\sqrt{\frac{4}{7}h_0h_{-2}} \tag{24}$$

Because Kalman filter is an optimal estimator, any other estimator will not be better than it. Therefore, Eq.(25) can be regarded as the limit of the frequency estimaton. This limit is related to the intrinsic noises in the clock, not the measurement equipment. In general speaking, the precision of frequency estimation is limited by the noise levels in the clock, which cannot be improved arbitarily by improving measurement.

VI. Conclussion

We have analyzed the effects of clock noises on the phase and frequency estimations, respectively. In phase estimation, the error can be reduced by properly selecting suitable smooth length *NT* and smaple interval *T*. But in frequency estimation, the error cannot be reduced arbitarily by means of improving estimator or measurement equipment. The precison of frequency is limited by the intrinsic noises in the clock.

Reference

[1] J. A. Barnes et al., "Characterization of Frequency Stability", IEEE Trans. on Instru. and Measu., Vol. IM-20, No. 2, 1971

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